Sampreshan
UGC CARE GROUP 1
ISSN: 2347-2979
https://sampreshan.info/
Vol. 17, Issue No. 1, March 2024

Comparative Analysis of Polynomial Zeroes within Prescribed Bounds

V.R.Patil and Peerzada Naveed Ahad
Department of mathematics, Arts, Science and Commerce collage
Chikhaldara, Amravati, Maharashtra 444807
Email: vrpascc@gmail.com and peerzadanaveed28@gmail.com

Abstract

This paper presents a comparative study of zeroes of two distinct polynomials under specified bounds. The investigation focuses on understanding the behavior of polynomial roots within predefined bounds and elucidating the implications of such constraints on their properties.

Keywords: Zeros of Polynomials, Companion Matrix, Determinant, Trace, Normal Matrix, Rank and Numerical Radius.

Introduction

Finding the zeros of polynomials is a traditional topic that has fascinated many mathematicians since Cauchy first proposed it. This subject has numerous applications in various branches of mathematics and remains an intriguing topic for both numerical and complex analysis. The companion of Frobenius Polynomials and matrix analysis are linked by a matrix. It is employed in the numerical approximation as well as the matrix method's location of polynomial zeros.
Since Cauchy was one of the first to contribute to the theory of a polynomial zero location, numerous other interested parties have explored this topic. Numerous publications propose different upper and lower bounds for the moduli of the zeros. Because of the subject's widespread applications in fields like signal processing, communication theory, and control theory, there is a constant need for ever-better solutions.

Definition 1.1

A monic polynomial is a single variable polynomial (that is, univariate polynomial) in which the leading coefficient (the non-zero coefficient of highest degree) is equal to 1,

Definition 1.2

Let $H(z) = z^n + b_n z^{n-1} + \cdots + b_2 z + b_1$

be a monic polynomial of degree $n \geq 2$ with complex coefficients $b_1, b_2, \dots, b_n$, where $b_1 \neq 0$. Then the Frobenius companion matrix of H is given by

$$C(H) = \begin{bmatrix} -b_n & -b_{n-1} & \dots & -b_2 & -b_1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Theorem 1.1 [1]

If M is any zero of $(z) = z^n + b_n z^{n-1} + \dots + b_2 z + b_1$, then by Abdurakhmanov:

$$|M| \le \frac{1}{2}\left(|b_n| + \cos\frac{\pi}{n} + \sqrt{\left(|b_n - \cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}\right) \quad (1)$$

Theorem 1.2 [4]

If M is any zero of $(z) = z^n + b_n z^{n-1} + \dots + b_2 z + b_1$, then by Kittaneh:

$$|M| \le \frac{1}{2}\left(|b_n| + \cos\frac{\pi}{n} + \sqrt{\left(|b_n - \cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}\right) \quad (2)$$

Theorem 1.4 [4]

If M is any zero of $H(z) = z^n + b_n z^{n-1} + \dots + b_2 z + b_1$, then

BK(another bound for Kittaneh):

$$|M| \le \frac{1}{2}\left(|b_n| + 1 + \sqrt{(|b_n| - 1)^2 + 4\sqrt{\sum_{j=1}^{n-1}|b_j|^2}}\right)$$

.

Theorem 1.5 [4]

In Frobenius Companion Matrix.

If M is any zero of

$$H(z) = z^n + b_n z^{n-1} + \dots + b_2 z + b$$

, then

1. $|M| \le \max\{|b_1|, 1 + |b_2|, 1 + |b_3|, \dots, 1 + |b_n|\}$
   $\le 1 + \max\{|b_1|, |b_2|, |b_3|, \dots, |b_n|\}$
   (Cauchy's bound)

2. $|M| \leq \max\{1, |b_1| + |b_2| + |b_3| + \cdots + |b_n|\}$
$\leq 1 + |b_1| + |b_2| + |b_3| + \cdots + |b_n|$
(Montel's bound)

3. $|M| \leq (1 + |b_1|^2 + |b|^2 + |b_3|^2 + \cdots + |b_n|^2)^{\frac{1}{2}}$
(Carmichael-Mason's bound)

Main Results

We represent the relation between the inequalities (1) and (2)

Theorem 2.1

Let $H(z) = z^n + b_n z^{n-1} + \cdots + b_2 z + b_1,\ b_1 \neq 0$ and z is any zero of P. then

$\frac{1}{\sqrt{k}} A.b \geq K.b$ for some $k > 0$ with $|k| > 1$.

Proof:

$$\frac{1}{2\sqrt{k}}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}\right)$$

$$\geq \frac{1}{2\sqrt{k}}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}\right)$$

If and only if

$$\frac{1}{2\sqrt{k}}|b_n| + \frac{1}{2}cos\frac{\pi}{n} + \frac{1}{2}\sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}$$

$$\geq \frac{1}{2}|b_n| + \frac{1}{2}cos\frac{\pi}{n} + \frac{1}{2}\sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}$$

If and only if

$$\frac{1}{2}\sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2} \qquad \text{(for } k = 1\text{)}$$

$$\geq \frac{1}{2}\sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}$$

If and only if

$$\sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2} \geq \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}$$

If and only if

$$\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2 \geq \left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2$$

If and only if

$$\left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2 \geq (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2$$

If and only if

$$1 + 2\sqrt{\sum_{j=1}^{n-1}|b_j|^2} + \sum_{j=1}^{n-1}|b_j|^2 \geq |b_{n-1}|^2 + 2|b_{n-1}| + 1 + \sum_{j=1}^{n-2}|b_j|^2$$

If and only if

$$2\sqrt{\sum_{j=1}^{n-1}|b_j|^2} \geq 2|b_{n-1}|$$

If and only if

$$\sum_{j=1}^{n-2}|b_j|^2 \geq 0$$

And hence, we see that

$$\sum_{j=1}^{n-2}|b_j|^2 \geq 0$$

always true.

So, the inequality (2) is better than then the inequality (1) because otherwise

$$\sum_{j=1}^{n-2}|b_j|^2 \leq 0$$

which is false.

Also, Kittaneh bound and Abdurakmanov bound aren't equal because if Kittaneh bound equals Abdurakmanov bound, then

$$\sum_{j=1}^{n-2}|b_j|^2 = 0$$

and hence $|b_j|^2 = 0$, which is false because $b_1 \neq 0$.

In the following example, we show that Kittaneh bound is better than Abdurakmanov bound.

Example 2.1

Let $h(z) = z^4 + 5$ then

$b_4 = 0, b_3 = 0, b_2 = 0, b_1 = 5\ and\ for\ k = 1.1 > 0$

So,

Abdurakmanov:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}\right) \leq 3.2153$$

Kittaneh:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}\right) \leq 2.9274$$

In the following example, we show that Kittaneh bound is better than Abdurakmanov bound.

Example 2.2

Let $h(z) = z^4 + 3z^3 + 10z^2 + 2z + 1$ then

$b_4 = 3, b_3 = 10, b_2 = 2, b_1 = 1\ and\ for\ some\ k = \frac{6}{5}$

So,

Abdurakmanov:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1}|b_j|^2}\right)^2}\right) \leq 3.0803$$

Kittaneh:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2}|b_j|^2}\right) \leq 2.9274$$

In the following example, we show that Kittaneh bound is better than Abdurakmanov bound.

Example 2.3

Let $h(z) = z^6 + 3z^5 + 4z^4 + 2z^2 + 3$ then

$b_6 = 3, b_5 = 4, b_4 = 0, b_3 = 2, b_2 = 0, b_1 = 3\ and\ for\ k = 4$

So,

Abdurakmanov:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |b_j|^2}\right)^2}\right) \leq 8.485$$

Kittaneh:

$$|M| \leq \frac{1}{2}\left(|b_n| + cos\frac{\pi}{n} + \sqrt{\left(|b_n| - cos\frac{\pi}{n}\right)^2 + (|b_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |b_j|^2}\right) \leq 5.194$$

References

[1] Abdurakhmanov, A. A. (1988). The geometry of the Hausdorff domain in localization problems for the spectrum of arbitrary matrices. Mathematics of the USSR-Sbornik, 59 (1), 39.

[2] Alhawari, M. O. H. A. M. M. A. D. (2005). New Estimate for the Numerical Radius of a Given Matrix, and Bounds for the Zeros of Polynomials. Editorial Advisory Board e, 16 (1), 90-95.

[3] Al-Hawari, M., & Brahmeh, R. New inequalities of the real parts of the zeros of polynomials.

[4] Kittaneh, F., & Shebrawi, K. (2007). Bounds for the zeros of polynomials from matrix inequalities–II. Linear and Multilinear Algebra, 55 (2), 147-158.

[5] Al-Hawari, M. (2003). LOCATION OF THE ZEROS OF POLYNOMIALS (Doctoral dissertation, University of Jordan).

[6] Al-Hawari, M., & Aldahash, A. A. (2013, April). New Inequalities for Numerical Radius of Hilbert Space Operator And New Bounds For The Zeros Of Polynomials. In Journal of Physics: Conference Series (Vol. 423, No. 1, p. 012013). IOP Publishing.

[7] Al-Hawari, M., & AL-Askar, F. M. (2013). SOME EXTENSION AND GENERALIZATION OF THE BOUNDS FOR THE ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS. International Journal of Pure and Applied Mathematics, 89 (4), 559-564.